Now, in view of Eq. (9), the stress function $\bar{T}_k$ becomes

$$\bar{T}_k = (1 + \alpha \sin\Theta)^{-2}(V^* + V_0) \tag{29}$$

All the forces, moments and strain components can be determined in terms of the stress function $\bar{T}_k$ by means of the formulas given in Ref. 1.

The power series appearing in the solution converges slowly, as $|\sin\Theta|$ approaches unity. However inequality (7) then becomes valid and the aforementioned usual asymptotic methods can be applied. Both solutions must be matched along lines θ const common to two adjacent parts of the shell.

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Flow of Combustion Gases through a Perforation in a Solid Propellant Grain

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Nomenclature

A = duct or perforation cross-sectional area; A_w is the duct wall surface area; A^* is the critical cross-sectional area ($M = 1$)
F = frictional term $\triangleq [(\gamma/2)K(M_0)^2]$
K = geometry factor, Fig. 2
M = Mach number = velocity$/(g_c\gamma RT)^{1/2}$, where g_c is gravitational constant and T is static temperature
P = pressure; P_s is stagnation pressure
R = mixture gas constant = $R_0 m_g^*/\mu_g$; R_0 is universal gas constant; m_g^* is gas phase mass fraction; μ_g is gas phase molecular weight
w = mass flow rate; $w^* \triangleq w/w_i$
γ = isentropic path exponent that is the ratio of the mixture specific heats
τ = shear stress; $\tau_w dA_w$ is wall drag force, Fig. 1
σ $\triangleq \gamma M^2$
$\phi(M)$ = $M\{[2/(\gamma + 1)][1 + (\frac{1}{2})(\gamma - 1)M^2]\}^{-(\gamma+1)/2(\gamma-1)}$ designated as A^*/A in standard gasdynamics table

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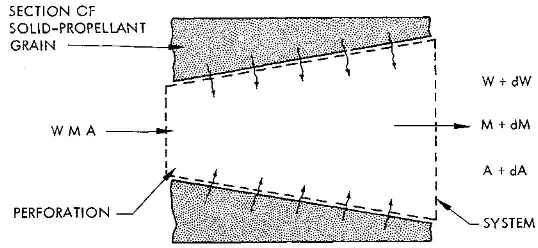

Fig. 1 Flow through a perforation in a solid propellant grain.

Subscripts

i, j = boundary condition
0 = value at smallest cross-sectional area at abrupt expansion or contraction, Fig. 2

Introduction

In the design of solid propellant grains for solid rockets, the flow of combustion products inside perforations in the grains must be considered. This involves the analysis of a flowing gas stream with mass being added to it due to the burning propellant (Fig. 1). This flowing gas stream may also contain a condensed phase, as aluminum oxide, because of the burning of a metallized propellant.

A special case of this problem, involving the steady one-dimensional flow of a perfect gas through a constant cross-sectional area duct with mass addition along the duct and with a constant stagnation temperature, has been treated by several authors.[1,2] Wimpress[1] has solved for the properties along the duct using a Mach number dependent variable as the independent variable in his solution. Wimpress includes the frictional effects due to sudden expansions which may occur at the end of a solid propellant grain. Price[2] has solved for the properties along the duct using a nondimensional mass flow rate as the independent variable. As pointed out by Price, this choice of independent variable is more convenient to use than a Mach number dependent variable.

The analysis to follow considers the steady one-dimensional equilibrium flow of a perfect gas and condensed phase mixture in a duct whose cross-sectional area varies with length and whose mass flow rate increases with length. An approximate solution that lends itself to hand calculations is given. Eddy mixing effects due to abrupt expansions or contractions are also considered.

Analysis

The following idealizations apply to the system shown in Fig. 1: 1) the flow is one-dimensional and steady; 2) the main stream stagnation temperature is constant (the system is adiabatic and the propellant flame temperature is constant); 3) the mass addition contributes negligible momentum rate in the axial direction; 4) the wall drag force $\tau_w dA_w$ is in the axial direction; 5) the two phases are in equilibrium, and the mixture properties are constant; and 6) the condensed

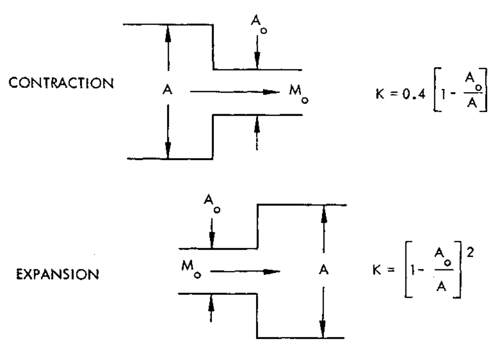

Fig. 2 Abrupt changes in perforation cross-sectional area.

phase occupies a negligible volume compared to that of the gas and the gas is calorically perfect.

Based on idealizations 5 and 6, the flowing two phase mixture can be treated as a single-phase calorically-perfect gas with properties γ and R as defined in the Nomenclature. The basic principles of mass, momentum, energy and definitions of Mach number, stagnation temperature, and stagnation pressure can be applied to the system of Fig. 1 as done in Ref. 3. The result is a series of differential equations coupled only in Mach number. However, only two of these differential equations need be solved (the equation for Mach number and any other one) to evaluate all the properties at any location from the given boundary conditions.

The two differential equations that are considered herein are

$$\frac{1 - M^2}{2[1 + (\frac{1}{2})(\gamma - 1)M^2]} \frac{dM^2}{M^2} = -\frac{dA}{A} + (1 + \gamma M^2)\frac{dw}{w} + \frac{\tau_w dA_w}{PA} \quad (1)$$

$$\frac{dP_s}{P_s} = -\gamma M^2 \frac{dw}{w} - \frac{\tau_w dA_w}{PA} \quad (2)$$

An approximate solution to Eq. (1) for $M < 1$ can be obtained after noting that Eq. (1) is exact except for the $(1 + \gamma M^2)$ coefficient on dw/w and possibly $\tau_w dA_w/PA$. Let $(1 + \gamma M^2) \triangleq (1 + \sigma)$, a constant over a Mach number interval, and let $\tau_w dA_w/PA = dF$, an exact differential over the same Mach number interval. (This last term is discussed below.) Then Eq. (1) is exact and can be integrated. The left-hand side of Eq. (1) is recognized as the derivative of the Mach number function

$$\frac{A^*}{A} = M\left[\left(\frac{2}{\gamma + 1}\right)\left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{-(\gamma+1)/2(\gamma-1)} \quad (3)$$

which relates the ratio of the critical cross-sectional area ($M = 1$) and any cross-sectional area A to the Mach number at A for isentropic flow of a perfect gas with a constant mass flow rate.[3] For convenience, this function is denoted as $\phi(M)$. Then the integration of Eq. (1) results in the expression

$$\frac{A_i}{A}\left(\frac{w}{w_i}\right)^{1+\sigma} \exp(\mp F) = \frac{\phi(M)}{\phi(M_i)} \quad (4)$$

where the negative sign corresponds to $(\ )_i$ downstream of $(\ )$. Since $(1 + \sigma)$ is not a strong function of M, the equation can be used over relatively large Mach number intervals as demonstrated below. The differential equation for stagnation pressure [Eq. (2)] is not exact, but depends on the path of M vs w. The integral of Eq. (2) is

$$\frac{P_s}{(P_s)_j} = \exp\left\{-\gamma \int_{w_j}^{w} \frac{M^2}{w}\, dw \mp F\right\} \quad (5)$$

where the negative sign corresponds to $(\ )_j$ upstream of $(\ )$. Because of the implicit nature of Eq. (4), the integral in Eq. (5) must be evaluated numerically.

The friction term F can be a result of wall shear stress or it can be a result of eddy mixing associated with abrupt cross-sectional area changes. For flow inside solid propellant grains where the cross-sectional area is varying gradually with length, the wall shear force can be neglected compared to the variation in momentum rates of the flowing stream, so that $F = 0$ for these cases. For abrupt cross-sectional area changes where flow separation or eddy mixing occurs simultaneously with mass addition, experimental data is needed to evaluate F. Since this data is lacking, available eddy mixing loss data for incompressible flow in a pipe must be used to estimate F.[4] Based on these data, the following is recommended:

$$F = (\gamma/2)[K(M_0)^2]$$

where K is given in Fig. 2.

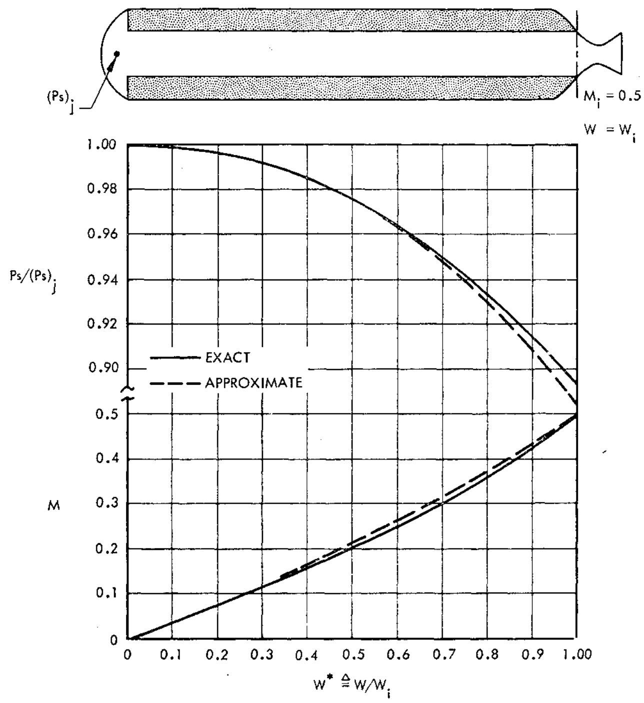

Fig. 3 Comparison of approximate solution with exact solution for a constant cross-sectional area application.

Discussion

The approximate solution for Mach number [Eq. (4)], the stagnation pressure relation [Eq. (5)], and standard relations involving properties for a perfect gas provide a means for calculating all the properties at any location inside a perforated solid propellant grain whose cross-sectional area is varying with length. The main advantage in the use of Eq. (4) is that it lends itself to hand calculations. The function of $\phi(M)$ is available in standard gasdynamics tables for various values of γ (i.e., Ref. 5).

The effect of eddy mixing due to abrupt cross-sectional area changes appears in the equation for Mach number as well as for stagnation pressure. However, the effect of F in Eq. (4) is usually small as compared to that of w or A, whereas in Eq. (5), the value of F is frequently about the same size as the integral.

A comparison can be made between the exact solution for the constant cross-sectional area case[2] and Eqs. (4) and (5) by applying them to a practical problem involving a constant cross-sectional area. Consider the solid rocket motor shown in Fig. 3. The Mach number M and $P_s/(P_s)_j$ vs w^* ($\triangleq w/w_i$) are shown in Fig. 3 for the exact solution and for the approximate solution [Eqs. (4) and (5)] with $(1 + \gamma M^2) = 1 + (1.19)(0.25)^2 = 1.074$. The approximate and exact solutions agree within 0.015 in Mach number and 0.010 in $P_s/(P_s)_j$.

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